

Comment on: Topological invariants, instantons, and the chiral anomaly on spaces with torsion

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In Riemann–Cartan spacetimes with torsion only its axial covector piece A couples to *massive* Dirac fields. Using renormalization group arguments, we show that besides the familiar Riemannian term only the Pontrjagin type four–form $dA \wedge dA$ does arise additionally in the *chiral anomaly*, but *not* the Nieh–Yan term d^*A , as has been claimed in a recent paper [PRD **55**, 7580 (1997)].

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I. INTRODUCTION

Quantum anomalies both in the Riemannian and in the Riemann–Cartan spacetimes were calculated previously using different methods, see e.g. [11,25]. However, recently [3] the completeness of these earlier calculations have been questioned which all demonstrated that the Nieh–Yan four–form [18] is irrelevant to the axial anomaly.

For the axial anomaly, we have a couple of distinguished features. Most prominent is its relation with the Atiyah–Singer index theorem. But also from the viewpoint of perturbative *quantum* field theory (QFT), the chiral anomaly has some features which signal its conceptual importance. For all topological field theories like BF-theories, Chern–Simons, and for all topological effects like the anomaly, the remarkable fact holds that the relevant invariants do not renormalize — higher order loop corrections do not alter the one-loop value of the anomaly, for example. The fact that the anomaly is stable against radiative corrections guarantees that it can be given a topological interpretation. For the anomaly, this is the Adler–Bardeen theorem, while other topological field theories are carefully designed to have, amongst other properties, vanishing beta functions. Another feature is finiteness: in any approach, the chiral anomaly as a topological invariant is a finite quantity.

In a spacetime with torsion, Chandia and Zanelli [3] argue that the Nieh–Yan (NY) four–form d^*A will add to

this quantity. As usual, they confront the fact that such a term, if it is generated at all, is ill-defined, independent of the regularization. In their case, they use a Fujikawa-type approach and propose to absorb the regulator mass in a rescaled vierbein.

However, there is a severe misunderstanding in the Ref. [3]. While there is no doubt that the NY term can be possibly generated, as demonstrated previously [19,25], this is not the end of the argument. In order to obtain a finite quantity, the tetrads have to be rescaled. While this might look as an innocent manipulation, this is not so. In rescaling the tetrad, the authors of Ref. [3] ignore the presence of renormalization conditions and the generation of a scale upon renormalization. Rescaling the tetrad would ultimately change the wave function renormalization Z -factor.

This factor creeps into the definition of the NY term at the quantum level, and thus a rescaling of the tetrad does not achieve the desired goals. This is not to be surprised: QFT demands a new Z -factor for the NY term, in sharp contrast to proper topological invariants at the quantum level, which remain unchanged under renormalization.

With no renormalization condition available for the NY term, and other methods obtaining it as zero, we can only conclude that the response function of the quantum field theory to a gauge variation (this is the anomaly) delivers no NY term. Or, saying it differently, its finite value is zero after renormalization.

II. GRAVITATIONAL CHERN–SIMONS AND PONTRJAGIN TERMS

In our notation, Clifford–algebra valued exterior forms [16], the constant Dirac matrices γ_α obeying $\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2o_{\alpha\beta}$ are saturating the index of the orthonormal coframe one–form ϑ^α and its Hodge dual $\eta^\alpha := *\vartheta^\alpha$ via $\gamma := \gamma_\alpha \vartheta^\alpha$ and $*\gamma = \gamma^\alpha \eta_\alpha$. In terms of the *connection* $\Gamma := \frac{i}{4} \Gamma^{\alpha\beta} \sigma_{\alpha\beta}$, the $SL(2, C)$ –covariant exterior derivative is given by $D = d + \Gamma \wedge$, where $\sigma_{\alpha\beta} = \frac{i}{2}(\gamma_\alpha \gamma_\beta - \gamma_\beta \gamma_\alpha)$ are the Lorentz generators entering also in the Clifford–algebra valued two–form $\sigma := \frac{i}{2} \gamma \wedge \gamma = \frac{1}{2} \sigma_{\alpha\beta} \vartheta^\alpha \wedge \vartheta^\beta$.

Differentiation of these independent variables leads to the Clifford algebra–valued two–forms of *torsion* $\Theta := D\gamma = T^\alpha \gamma_\alpha$ and *curvature* $\Omega := d\Gamma + \Gamma \wedge \Gamma = \frac{i}{4} R^{\alpha\beta} \sigma_{\alpha\beta}$ of Riemann–Cartan (RC) geometry.

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The *Chern–Simons* (CS) term [7] for the Lorentz connection $C_{\text{RR}} := -\text{Tr}(\Gamma \wedge \Omega - \frac{1}{3}\Gamma \wedge \Gamma \wedge \Gamma)$ and its corresponding Pontrjagin term $dC_{\text{RR}} = -\text{Tr}(\Omega \wedge \Omega) = \frac{1}{2}R^{\alpha\beta} \wedge R_{\alpha\beta}$ have the familiar form. Since the coframe is the translational part of the Cartan connection [7], there arises also the *translational* CS term [15]

$$C_{\text{TT}} := \frac{1}{8\ell^2} \text{Tr}(\gamma \wedge \Theta) = \frac{1}{2} (C_{\text{RR}} - \hat{C}_{\text{RR}}) \quad (2.1)$$

which is related to the Nieh–Yan four-form [18]:

$$dC_{\text{TT}} = \frac{2}{\ell^2} (T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta). \quad (2.2)$$

A fundamental length ℓ unavoidably occurs here for dimensional reasons. This can be also motivated by a de Sitter type [6] approach, in which the $sl(5, R)$ -valued connection $\hat{\Gamma} = \Gamma + (1/\ell)(\vartheta^\alpha L^4_\alpha + \vartheta_\beta L^\beta_4)$ is expanded into the dimensionless linear connection Γ plus the coframe $\vartheta^\alpha = e_i^\alpha dx^i$ with canonical dimension [length]. The corresponding CS term \hat{C}_{RR} splits via $\hat{C}_{\text{RR}} = C_{\text{RR}} - 2C_{\text{TT}}$ into the linear one and that of translations, see the footnote 31 of Ref. [7]. This relation has recently been “recovered” by Chandia and Zanelli [3].

The one-form of *axial vector torsion*

$$A := \frac{1}{4} \text{Tr}(\tilde{\gamma} \lrcorner^* \Theta) = \frac{1}{4} {}^* \text{Tr}(\gamma \wedge \Theta) = {}^*(\vartheta^\alpha \wedge T_\alpha) \quad (2.3)$$

is a *conformal invariant* under the combined transformation of *classical* Weyl rescalings of the coframe, in contrast to ${}^*A = -2\ell^2 C_{\text{TT}}$, cf. Eqs. (3.14.1,9) of Ref. [7].

III. DIRAC FIELDS IN RIEMANN–CARTAN SPACETIME

The Dirac Lagrangian is given by the manifestly *Hermitian* four-form

$$\begin{aligned} L_{\text{D}}(\gamma, \psi, D\psi) &= \frac{i}{2} \{ \bar{\psi}^* \gamma \wedge D\psi + \overline{D\psi} \wedge {}^* \gamma \psi \} + {}^* m \bar{\psi} \psi \\ &= L(\gamma, \psi, D\lrcorner \psi) - \frac{1}{4} A \wedge \bar{\psi} \gamma_5 {}^* \gamma \psi, \end{aligned} \quad (3.1)$$

for which $\bar{\psi} := \psi^\dagger \gamma_0$ is the Dirac adjoint and ${}^*m = m\eta$ the mass term, cf. [16].

The decomposed Lagrangian (3.1) leads to the following form of the Dirac equation

$$i {}^* \gamma \wedge \widetilde{D} \psi + {}^* m \psi = i {}^* \gamma \wedge \left[D\lrcorner + \frac{i}{4} m \gamma + \frac{i}{4} A \gamma_5 \right] \psi = 0 \quad (3.2)$$

in terms of the Riemannian connection $\Gamma\lrcorner$ with $D\lrcorner \gamma = 0$ and the *irreducible* piece (2.3) of the torsion. Hence, in a RC spacetime a Dirac spinor does only feel the *axial torsion* one-form A . This can also be seen from the

identity (3.6.13) of Ref. [7] which specializes here to the “on shell” commutation relation

$$[\widetilde{D}, \widetilde{D}] = \Omega\lrcorner + \frac{i}{4} \gamma_5 dA - \frac{i}{8} m^2 \sigma. \quad (3.3)$$

In contrast to Ref. [3], Eq. (27), there arise in (3.3) no tensor or vector pieces of the torsion, because our operator \widetilde{D} in (3.2) is the only possible result from the Lagrangian (3.1), which is *Hermitian* as required by QFT.

From the Dirac equation (3.2) and its adjoint one can readily deduce the well-known “classical axial anomaly” $dj_5 = d(\frac{1}{3}\bar{\psi}\sigma \wedge \gamma\psi) = 2miP = 2mi\bar{\psi}\gamma_5\psi$ for *massive* Dirac fields also in a RC spacetime. If we restore chiral symmetry in the limit $m \rightarrow 0$, this leads to *classical conservation law* of the axial current for massless Weyl spinors, or since $dj = 0$, equivalently, for the *chiral current* $j_\pm := \frac{1}{2}\bar{\psi}(1 \pm \gamma_5) {}^* \gamma \psi = \bar{\psi}_{\text{L,R}} {}^* \gamma \psi_{\text{L,R}}$.

The Einstein–Cartan–Dirac (ECD) theory of a gravitationally coupled spin $\frac{1}{2}$ fermion field provides a *dynamical* understanding of the axial anomaly on a classical (i.e., not quantized) level. From Einstein’s equations $-(1/2)\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} = \ell^2 \Sigma_\alpha$ and the purely algebraic *Cartan relation* $-(1/2)\eta_{\alpha\beta\gamma} \wedge T^\gamma = \ell^2 \tau_{\alpha\beta} = -(\ell^2/4)\eta_{\alpha\beta\gamma\delta} \bar{\Psi} \gamma_5 \gamma^\delta \Psi \eta^\gamma$ one finds [16,17]

$$dj_5 \cong 4dC_{\text{TT}} = \frac{2}{\ell^2} (T^\alpha \wedge T_\alpha + R_{\alpha\beta} \wedge \vartheta^\alpha \wedge \vartheta^\beta) \quad (3.4)$$

which establishes a link to the NY four form [18], but only for *massive* fields [17].

However, if we restore chiral invariance for the Dirac fields in the limit $m \rightarrow 0$, we find within the dynamical framework of ECD theory that the NY four-form tends to zero “on shell”, i.e. $dC_{\text{TT}} \cong (1/4)dj_5 \rightarrow 0$.

This is consistent with the fact that a Weyl spinor does not couple to torsion at all, because the remaining axial torsion A becomes a *lightlike* covector, i.e. $A_\alpha A^\alpha \eta = A \wedge {}^*A \cong (\ell^4/4) {}^*j_5 \wedge j_5 = 0$. Here we implicitly assume that the light-cone structure of the axial covector *j_5 is not spoiled by quantum corrections, i.e. that no “Lorentz anomaly” occurs as in $n = 4k + 2$ dimensions [14].

IV. CHIRAL ANOMALY IN QFT

When quantum field theory (QFT) is involved, other boundary terms may arise in the *chiral anomaly* due to the non-conservation of the axial current, cf. [26,8].

Now, to approach the anomaly in the context of spacetime with torsion, we will proceed by switching off the curvature and concentrate on the last term in the decomposed Dirac Lagrangian (3.1).

Then, this term can be regarded as an *external* axial covector A (in view of (3.3) without Lorentz or “internal” indices) coupled to the axial current j_5 of the Dirac field in an *initially flat* spacetime. By applying the result (11–225) of Itzykson and Zuber [9], we find that only the term $dA \wedge dA$ arises in the axial anomaly, but *not* the

NY type term $d^*A \sim dC_{\text{TT}}$ as was recently claimed [3]. After switching on the curved spacetime of Riemannian geometry, we finally obtain for the axial anomaly

$$\langle dj_5 \rangle = 2m\langle iP \rangle + \frac{1}{24\pi^2} \left[\text{Tr}(\Omega^\Omega \wedge \Omega^\Omega) - \frac{1}{4} dA \wedge dA \right]. \quad (4.1)$$

Besides other perturbative methods as point-splitting, there is the further option to use *dimensional regularization*. If one adopts the γ_5 scheme of Ref. [12], one immediately concludes that only the result (4.1) can appear. The only effect of the γ_5 problem is the replacement of the usual trace by a non-cyclic linear functional. The anomaly appears as the sole effect of this non-cyclicity. There is no room for other sources of non-cyclicity apart from the very fermion loops which produce the result (4.1). The whole effect of non-cyclicity is to have an operator Δ , $\Delta^2 = 0$, and the anomaly is in the image modulo the kernel of Δ , which summarizes the fact that in this γ_5 scheme no other anomalous contributions are possible beside (4.1).

But at this stage we have not discussed the possibility of a contorted spacetime which *cannot* be *adiabatically* deformed to the torsion-free case. In such a case it has been argued [3] that the boundary term dC_{TT} occurs, multiplied by a factor M^2 . This factor M^2 corresponds to a regulator mass in a Fujikawa type approach. For instance, in the heat kernel approach, the first nontrivial terms [19,25], which potentially could contribute to the axial anomaly, read

$$\begin{aligned} \text{Tr}(\gamma_5 K_2) &= -d^*A, & \mathcal{K} &= {}^* \widetilde{D} \wedge {}^* \widetilde{D}^* A \\ \text{Tr}(\gamma_5 K_4) &= \frac{1}{6} \left[\text{Tr}(\Omega^\Omega \wedge \Omega^\Omega) - \frac{1}{4} dA \wedge dA + d\mathcal{K} \right]. \end{aligned} \quad (4.2)$$

However, there is an essential difference in the physical dimensionality of the terms K_2 and K_4 . Whereas in $n = 4$ dimensions the Pontrjagin type term K_4 is dimensionless, the term $K_2 \sim 2\ell^2 dC_{\text{TT}}$ carries dimensions. It can be consistently absorbed in a counterterm, and thus *discarded* from the final result for the anomaly.

This is also in agreement with the analysis in [2] where, in the framework of string theory, the chiral anomaly in the presence of torsion had a smooth adiabatic limit to the case of vanishing torsion.

In contrast, in Ref. [3] it is argued that such contributions can be maintained by absorbing the divergent factor in a rescaled coframe $\vartheta^\alpha := M\vartheta^\alpha$ and propose to consider the Wigner–Inönü contraction $M \rightarrow \infty$ in the de Sitter gauge approach [6], with $M\ell$ fixed.

Apart from the fact that this would change also the dimension of ψ , in order to retain the physical dimension $[\hbar]$ of the Dirac action, there are several points which seem unsatisfactory in such an argument:

1. As the difference (2.1) of two Pontrjagin classes, the term dC_{TT} is a topological invariant after all. Now,

it is actually *not* this term which appears as the torsion-dependent extra contribution to the anomaly, but more precisely $-d^*A = 2\ell^2 dC_{\text{TT}}$. Thus, measuring its proportion in units of the topological invariant dC_{TT} , we find that it vanishes when we consider the proposed limit $M \rightarrow \infty$, keeping $M\ell$ constant.

2. Instead of rescaling the vierbein, it is consistent to compensate the ill-defined term by a counterterm. This implies that consistently a renormalization condition can be imposed which guarantees the anomaly to have the value (4.1). Even if one renders this extra term finite by a rescaling as in Ref. [3], one has to confront the fact that a (finite) renormalization condition can be imposed which settles the anomaly at this value. Further, if one were to keep this extra finite term, it would be undetermined, and is thus not related to the anomaly at all. Also, on-shell renormalization conditions adjust the wave function renormalization of the fermion propagator to have unit residuum at the physical mass. Any rescaling of the tetrad cannot dispense for the fact that the NY term needs renormalization by itself, as it is proven by the very calculation of [3].

3. From a *renormalization group* point of view, it is the scaling of the coupling which determines the scaling of the anomaly (regarded as a Green's function), a property which is desperately needed to maintain the validity, e.g., of the proof of the Adler–Bardeen theorem. Or, to put it otherwise, an anomaly is stable against radiative corrections for the reason that such corrections are compensated by a renormalization of the coupling. While, on the other hand, the topological invariant of Ref. [3] has no such property, its interpretation as an anomaly seems dubious to us. The only consistent way out is to impose a renormalization condition which adjust the finite value of the NY term to be zero, which is patently stable under radiative corrections.

4. Finally, it is well-known that usually the appearance of a chiral anomaly is deeply connected with the presence of a *conformal anomaly* [21,5,10]. This makes sense: usually, conformal invariance is lost due to the (dynamical) generation of a scale. But this is the very mechanism which destroys chiral invariance as well. Thus, one would expect any argument to fail trying to combine strict conformal invariance with a chiral anomaly.

V. CONCLUSIONS

Our conclusion is that the NY term dC_{TT} does NOT contribute to the *chiral anomaly* in $n = 4$ dimensions, neither classically nor in quantum field theory, in sharp contrast to Ref. [3]. We once more stress the interrelation between the scale and chiral invariance [21,5,10]. Since renormalization amounts to a continuous scale deformation, only the Riemannian part of the Pontrjagin term contributes to the topology of the chiral anomaly.

The result of Chandia et al. is very different in spirit

from a typical anomaly, where, in pQFT, the relevant Green's function is UV-finite and only implicitly depends on the scale via the coupling. This contrasts the fact that the Chandia et al. term would depend *explicitly* on that scale.

Since the A is *not* a gauge field, one can also legitimately absorb the contribution from the axial torsion covector A into the redefined *gauge-invariant* physical current $\hat{j}_5 := j_5 + (1/96\pi^2)A \wedge dA + M^2 *A$, where the last term depends explicitly on the regulator mass M . It may arise from the counterterm $M^2 A \wedge *A$ to the Dirac Lagrangian (3.2). Then, due to

$$\begin{aligned} \langle \hat{d}j_5 \rangle &= \langle dj_5 \rangle + (1/96\pi^2)dA \wedge dA + M^2 d*A \\ &= 2m\langle iP \rangle + (1/24\pi^2)Tr(\Omega^{\{\}} \wedge \Omega^{\{\}}), \end{aligned} \quad (5.1)$$

only the *Riemannian* contribution remains for the axial anomaly of this new *physical* current. A consistent way to avoid regularization problems for $M \rightarrow \infty$ is to assume that the “photon” A is transverse, i.e. $d*A = 0$, which is just the vanishing of the NY term.

One would surmise that in $n = 2$ dimensional models only the term $d*A$ survives in the heat kernel expansion (4.2), since it then has the correct dimensions. However, it is well-known [7] that in 2D the axial torsion A vanishes identically.

In general, the Pontrjagin topological invariant, as an integral over a *closed* four-form, depends also upon the second fundamental form of the embedding of the boundary ∂M into M . In Ref. [20] this is generalized to spaces with torsion supporting our view that the index shall be independent of regulator masses, hence excluding contributions from the NY term. Since the integral $\int_M d*A \equiv \int_{\partial M} *A$ vanishes *identically* for manifolds without boundary, the NY invariant would occur only for non-closed manifolds, anyhow.

A situation where torsion is indeed realized in a *discontinuous* manner arises for the cosmic string solution within the EC theory [22,1,24]. We have shown in detail in Ref. [17] that the NY term (2.2) *vanishes identically* for this example of a spinning cosmic string exhibiting a *torsion line defect*. The instantons of Ref. [3] with non-vanishing NY term are clearly detached from an Einstein-type dynamics, and also the recent analysis in Ref. [23] fails to substantiate the presence of the NY term.

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